

Spanning trees in random satisfiability problems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2006 J. Phys. A: Math. Gen. 39 4901

(<http://iopscience.iop.org/0305-4470/39/18/008>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.104

The article was downloaded on 03/06/2010 at 04:26

Please note that [terms and conditions apply](#).

Spanning trees in random satisfiability problems

A Ramezani¹ and S Moghimi-Araghi²

¹ Institute for Advanced Studies in Basic Sciences, Zanjan 45195-1159, Iran

² Department of Physics, Sharif University of Technology, PO Box 11365-9161, Tehran, Iran

E-mail: ramezani@iasbs.ac.ir and samanimi@sharif.edu

Received 21 November 2005, in final form 1 March 2006

Published 19 April 2006

Online at stacks.iop.org/JPhysA/39/4901

Abstract

Working with tree graphs is always easier than with loopy ones and spanning trees are the closest tree-like structures to a given graph. We find a correspondence between the solutions of random K -satisfiability problem and those of spanning trees in the associated factor graph. We introduce a modified survey propagation algorithm which returns null edges of the factor graph and helps us to find satisfiable spanning trees. This allows us to study organization of satisfiable spanning trees in the space spanned by spanning trees.

PACS numbers: 02.50.-r, 75.10.Nr, 64.60.Cn

1. Introduction

Understanding the nature of complexity in NP-hard optimization problems is an important goal in the study of complex systems [1, 2]. Most of optimization problems are indeed a constraint satisfaction problem and satisfiability problem is a prototype of such problems [3, 4]. In a satisfiability problem one looks for a configuration of logical variables that satisfies a number of logical constraints (clauses). A clause is in turn disjunction of a few logical variables. In random K -satisfiability (K -SAT) problem a clause involves K randomly selected variables each negated with a given probability. Increasing the number of clauses per variables, these problems display a phase transition from satisfiable to unsatisfiable phase [1, 5]. It is around this SAT–UNSAT transition that the typical complexity of the problem raises rapidly. In fact the transition is preceded by a clustering of solutions in the configuration space, a phenomenon known as replica symmetry breaking in physics literatures [6, 7]. There are a few analytic methods and algorithms, such as replica and cavity methods and the survey propagation algorithm [8–10], that deal with the behaviour of random satisfiability problems.

Here we are going to look at the problem from a rather geometrical point of view. We know that there are efficient algorithms such as WalkSAT and belief propagation [10–12] that can efficiently solve a random K -SAT problem possessing a tree-factor graph; a factor graph is a bipartite graph representation of constraint satisfaction problems. We also know that

spanning trees are the closest tree structures to a given graph. Then it is interesting to find the relation between the solutions of a factor graph and the solutions of its spanning trees. This relation could result in the reduction of the original problem to easier ones.

In this paper we show that there is indeed a correspondence between the solutions of a random K-SAT problem and those of spanning trees in the associated factor graph. In principle to check that a random K-SAT problem is satisfied or not one has to check the satisfiability of all the spanning trees. The main problem is that in general the number of spanning trees grows exponentially with the size of graph. Moreover, we will see that even when we are in the SAT phase almost all randomly generated spanning trees are unsatisfiable. To get around this problem we assign a weight to each edge of the factor graph. For an edge, its weight is taken as the probability that the edge to be present in the graph. Then we suggest an ansatz that relates the above probability to the survey of warnings sent along the edge and run survey propagation algorithm. This algorithm converges with a nonzero probability in the SAT phase and we find that in this case all maximum spanning trees (which have the maximum sum of weights on their edges) are satisfied. In other words, the above algorithm defines null edges of the factor graph.

The structure of the paper is as follows: in section 2 we define the problem more precisely and illustrate the relation between a factor graph and its spanning trees. We briefly deal with the satisfiability of randomly generated spanning trees in section 3. In section 4 we modify the survey propagation algorithm to generate satisfiable spanning trees. Section 5 includes our concluding remarks.

2. From factor graph to its spanning trees

A random satisfiability problem is defined as follows: we take N logical variables $x_i \in \{0, 1\}$. Then we construct a formula F of M clauses joined to each other by logical AND. Each clause contains a number of randomly selected logical variables. In random K-SAT problem each clause has a fixed number of K variables. These variables, which join to each other by logical OR, are negated with probability p and appear as such with probability $1 - p$. For example, $F := (\bar{x}_2 \vee x_4) \wedge (\bar{x}_3 \vee x_2) \wedge (x_1 \vee x_3)$ is a 2-SAT formula with three clauses and four logical variables. A solution of F is a configuration of logical variables that satisfy all clauses.

The relevant parameter that determines the satisfiability of F is $\alpha := M/N$. In the thermodynamic limit ($N, M \rightarrow \infty$ and $\alpha \rightarrow \text{const}$) F is satisfied with probability 1 as long as $\alpha < \alpha_c$. Moreover, it has been found that for $\alpha_d < \alpha < \alpha_c$ the problem is in the hard-SAT phase [9]. At α_d we have a dynamical phase transition associated with the break down of replica symmetry. Assuming one-step replica symmetry breaking, one obtains $\alpha_d \simeq 3.92$ and $\alpha_c \simeq 4.26$ for random 3-SAT problems with $p = 1/2$ [9]. Although this approximation seems to be exact near the SAT-UNSAT transition, but it fails close to the dynamical transition where higher order replica symmetry breaking solutions are to be used [13, 14]. The above approximation allows us to compute the configurational entropy or complexity of the problem. The complexity, which is logarithm of the number of clusters in the solution space, takes abruptly a nonzero value at α_d and decreases to zero at α_c .

The factor graph is a bipartite graph of variable nodes and function nodes (clauses). The structure of this graph is completely determined by an $M \times N$ matrix with elements $J_{a,i} \in \{0, +1, -1\}$; $J_{a,i} = +1$ if the clause a contains x_i , it is equal to -1 if \bar{x}_i appears in a and otherwise $J_{a,i} = 0$.

Now consider a random K-SAT problem and the associated factor graph. In the SAT phase the problem has some solutions that satisfy all the clauses. The factor graph has a number of spanning trees which have their own solutions. These solutions can be easily obtained by

WalkSAT algorithm or some other local search algorithms [15]. Note that a spanning tree could have no solution even when the original problem is satisfied. Here we show that (i) Each solution of a spanning tree is the solution of the original graph too. And (ii) Each solution of the original graph is the solution of at least one of the spanning trees.

Note that in a spanning tree each function node is connected to a smaller number of variable nodes than in the original graph. Thus if a configuration of variable nodes satisfies a spanning tree it should also satisfy the original problem.

To verify the second statement we start with a solution of the problem and try to construct a spanning tree that is also satisfied by this solution. To this end we have to remove some of the edges in the graph until no cycle remains in the factor graph. During this process we should, of course, respect the graph connectedness and more importantly satisfaction of all the function nodes. Therefore, at each step we consider a cycle of the graph and remove one of the null edges along this cycle. A null edge is an edge that its removal does not change the number of satisfied clauses. This removal also preserves the graph connectedness. Note that a null edge at a given stage may not be a null one in the next stages.

To complete the proof we should show that for a solution of the factor graph, any cycle in the graph has at least one null edge. First note that each function node which lies on a cycle has at least two neighbours. Since we are dealing with a solution then at least one of its neighbouring variable nodes satisfies the function node. In this case at least one of the edges along the cycle will be a null edge and we can freely remove it from the graph.

Indeed for any solution, a cycle would have a large number of null edges which result in a large degree of freedom in constructing the desired spanning tree. The number of null edges along a cycle would be of order of the number of function nodes in the cycle. It means that an arbitrary solution appears in a large number of spanning trees.

Suppose that we have a satisfiable spanning tree. Then we can use a local search algorithm to find its solutions. Note that for a tree factor graph there is no frustration in the problem and one can safely assume all the solutions in the same cluster. Indeed, warning propagation algorithm [12] which is based on this assumption gives exact results for tree factor graphs. Thus we expect that the solutions of a satisfiable spanning tree lay in the same cluster of solutions. In the thermodynamic limit, it means that for any two solutions, there is at least one path in the solution space that connects them to each other. The path starts from one of the solutions and reach the other one by successive steps of finite lengths. Since the solutions also satisfy the original problem, we may conclude that they lay in the same cluster of the original problem too.

Please note that one could use the same arguments (as given in this section) to show that any configuration of variables that violates E clauses in the original factor graph also appears in at least one of the spanning trees. Therefore, the state with minimum E (the ground state) in the original factor graph, has also the minimum E among the ground states of all the spanning trees.

3. Satisfiability of spanning trees

There is an algorithm that uses a simple random walker on the original factor graph and generates (with a uniform measure) a spanning tree [16]. Using this algorithm we generated up to 10^7 spanning trees of the factor graph of a random 3-SAT problem with $N = 100$. Belief propagation algorithm can be used to compute the entropy (logarithm of the number of solutions) of each spanning tree. We found that when $p = 0$ both the average entropy and its dispersion decrease exponentially with α . However, if we increase p slightly, we find that almost all the randomly generated spanning trees have no solution. For instance, with $\alpha = 4$

and $p = 0.1$, there is almost no satisfiable spanning tree among 10^7 randomly generated ones. In this case, in an average about 25 clauses are not satisfied. When we increase p to 0.5, the average number of unsatisfied clauses in a randomly generated spanning tree increases to 100, whereas the original problem is satisfiable. We found that any naive attempt, like the generalization of simple random walk procedure to a biased one, fails in generating satisfiable spanning trees. These observations show that finding satisfiable spanning trees is not an easy task. In the following we look for a more sophisticated way to solve this problem. We will restrict ourselves to random 3-SAT problems in the most random case, i.e. $p = 1/2$.

4. Generating satisfiable spanning trees

In section 2 we used the notion of null edges to construct spanning trees that are satisfied by a given solution. It suggests that some of the edges can be removed from the factor graph with no effect on the number of satisfied clauses. But how can we distinguish these null edges? Suppose that we remove some edges and run the survey propagation algorithm [10]. If the problem is in the SAT phase, the algorithm converges with a nonzero probability and for each edge returns a survey of warnings $\eta_{a \rightarrow i}$. The survey $\eta_{a \rightarrow i}$ is the probability that in an arbitrary cluster of solutions, clause a sends a warning to variable i . This warning enforces i to take a value which satisfies a . If the problem is in the UNSAT phase, then the algorithm may still converge but it results to a negative value for the complexity. In the following we will work with a version of survey propagation algorithm in which we assign zero probability to contradictory messages. Then the algorithm can end with a set of contradictory surveys. In this case a variable receives surveys that suggest different values for it and taking any value results in at least one unsatisfied clause.

Let us define $w_{a,i}$ as the probability that edge (a, i) to be present in the factor graph. Then we run the survey propagation algorithm taking these probabilities into account. It means that when function node a sends a message to its variable node i , it gives more weights to the neighbours having a higher probability of being present in the graph. More precisely, $\eta_{a \rightarrow i}$ is given by

$$\eta_{a \rightarrow i} = \prod_{j \in V(a) - i} [w_{a,j} P_a^u(j) + 1 - w_{a,j}], \quad (1)$$

where $P_a^u(j)$ is the probability that variable j cannot satisfy clause a . We also denote by $V(a)$ the set of variables belong to clause a and by $V(i)$ the set of clauses that variable i contributes in. In survey propagation algorithm $P_a^u(j)$ is given by [10]

$$P_a^u(j) = \frac{\Pi_{j \rightarrow a}^u}{\Pi_{j \rightarrow a}^s + \Pi_{j \rightarrow a}^0 + \Pi_{j \rightarrow a}^u}, \quad (2)$$

where

$$\begin{aligned} \Pi_{j \rightarrow a}^0 &= \prod_{b \in V(j) - a} (1 - w_{b,j} \eta_{b \rightarrow j}), \\ \Pi_{j \rightarrow a}^u &= \left[1 - \prod_{b \in V_a^u(j)} (1 - w_{b,j} \eta_{b \rightarrow j}) \right] \prod_{b \in V_a^s(j)} (1 - w_{b,j} \eta_{b \rightarrow j}), \\ \Pi_{j \rightarrow a}^s &= \left[1 - \prod_{b \in V_a^s(j)} (1 - w_{b,j} \eta_{b \rightarrow j}) \right] \prod_{b \in V_a^u(j)} (1 - w_{b,j} \eta_{b \rightarrow j}). \end{aligned} \quad (3)$$

Here $V_a^s(j)$ denotes to the set of clauses in $(V(j) - a)$ that variable j appears in them as it appears in clause a , see figure 1. The remaining set of clauses are denoted by $V_a^u(j)$. We

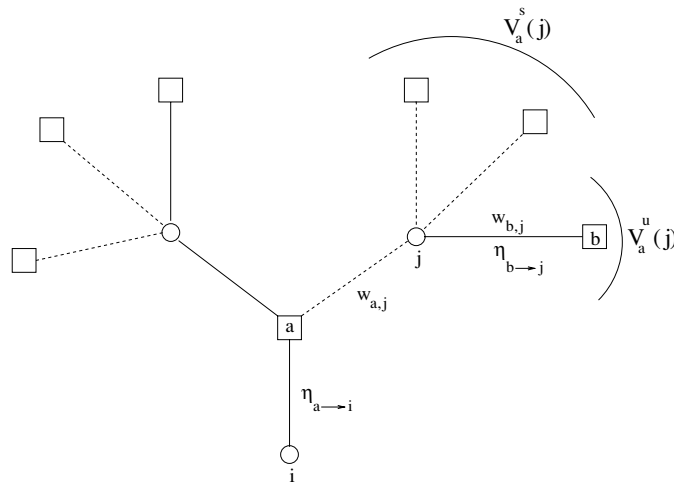


Figure 1. The survey $\eta_{a \rightarrow i}$ is determined by the set of surveys $\eta_{b \rightarrow j}$ and weights $w_{a,j}, w_{b,j}$. In this factor graph squares and circles represent clauses and variables, respectively. A dashed line shows that the negated variable appears in the clause.

should also give a way to determine $w_{a,i}$. Note that we expect a positive correlation between $w_{a,i}$ and $\eta_{a \rightarrow i}$. If the survey is large, the edge is necessary for satisfaction of the clause a and in the opposite case its presence is not essential. Our suggested ansatz for the weights is

$$w_{a,i} = [\eta_{a \rightarrow i}]^\mu, \tag{4}$$

where $\mu \geq 0$ controls the number of removed edges. For each instance of the problem we start by a random initialization for η 's and w 's. In each iteration we select all of the edges in a random way. When an edge is selected we first update $\eta_{a \rightarrow i}$ and then $w_{a,i}$ according to equations (1)–(4). We expect that for appropriate values of μ the above algorithm converges (without contradictory surveys) as long as we are in the SAT phase. In practice we define a maximum t_{\max} for the number of iterations t . Thus for a given μ it is possible not to reach a convergence in $t < t_{\max}$. We define P_{conv} as the probability that a randomly generated instance of the problem converges in a number of iterations less than t_{\max} . In our numerical simulations we will take $t_{\max} = 1000$ and the limit of convergence is taken $\epsilon = 10^{-3}$. Note that when $\mu = 0$ we recover the usual survey propagation algorithm on the original factor graph. In this situation the algorithm always converges and $P_{\text{conv}} = 1$. Increasing μ means that we are removing some of the edges and so we expect more time to reach a convergence. Thus $P_{\text{conv}}(\mu)$ would be a decreasing function of its argument. We find that its behaviour depends on α , figure 2. In fact, P_{conv} diminishes more rapidly for greater values of α .

Suppose that our algorithm converges. Then we expect that a spanning tree with maximum sum of weights for its edges, i.e. a maximum spanning tree, be a satisfiable one. When the algorithm converges and returns the w 's we can construct a maximum spanning tree as follows: starting from a randomly selected node in the original factor graph we first find the maximum weight among the edges that connect it to the other nodes. Then we list the edges having a weight in the ϵ -neighbourhood of the maximum one and add randomly one of them to the new factor graph. If we repeat the addition of edges $N + M - 1$ times we obtain a spanning tree factor graph which is a maximum spanning tree. Note that taking a nonzero interval to define the edges of maximum weight at each step, along with the randomness in choosing one of them,

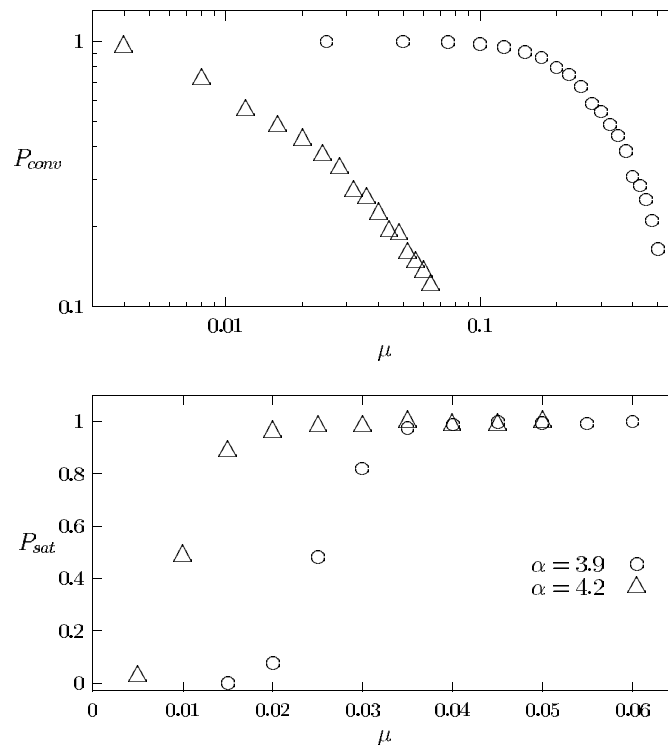


Figure 2. Convergence probability (P_{conv}) and satisfiability probability of a maximum spanning tree (P_{sat}) versus μ for a random 3-SAT problem with $N = 1000$ and $p = 1/2$. The data for P_{conv} obtained from 2000 realizations of the problem. To obtain P_{sat} , we randomly generated 20 maximum spanning trees in 150 satisfiable realizations.

allow us to construct a large number of maximum spanning trees. Let us define $P_{sat}(\mu)$ as the probability that a maximum spanning tree be satisfiable if the algorithm converges. The best value of μ is one that for which P_{sat} takes its maximum value. In figure 2 we also show $P_{sat}(\mu)$ for two values of $\alpha = 3.9, 4.2$. As the figure shows $P_{sat} \simeq 1$ when $\mu > 0.04$ ($\alpha = 3.9$) and $\mu > 0.025$ ($\alpha = 4.2$). However, as stated above, by increasing μ the convergence probability decreases. Thus the optimal μ for $\alpha = 3.9, 4.2$ will be around 0.04 and 0.025, respectively. The optimal μ decreases with α . It means that as we get closer to the SAT-UNSAT transition we have to give more importance to the surveys. To see the finite size effects we run the algorithm for larger problem sizes $N = 2000, 4000$. The results for P_{conv} and P_{sat} at $\alpha = 4.2$ have been shown in figure 3. The convergence probability decreases more rapidly with μ as N increases. That is we have to pay more computational efforts to get a converged situation. On the other side, by increasing N the satisfiability probability enhances for smaller values of μ and gets more rapidly its saturation value. We hope that this behaviour of P_{sat} compensate the decrease in P_{conv} for larger problem sizes. As figure 3 shows the changes in the quantities are significant and we have a strong size dependence for the small values of N studied here.

Finally, let us look at the statistics of w 's and the organization of maximum spanning trees in the space spanned by spanning trees.

We observe that if $P_{sat} \simeq 1$, then the weight distribution of the edges, $P(w)$, is almost sum of a few delta peaks. Figure 4 displays this behaviour of $P(w)$ for $\alpha = 3.9, 4.2$. In both

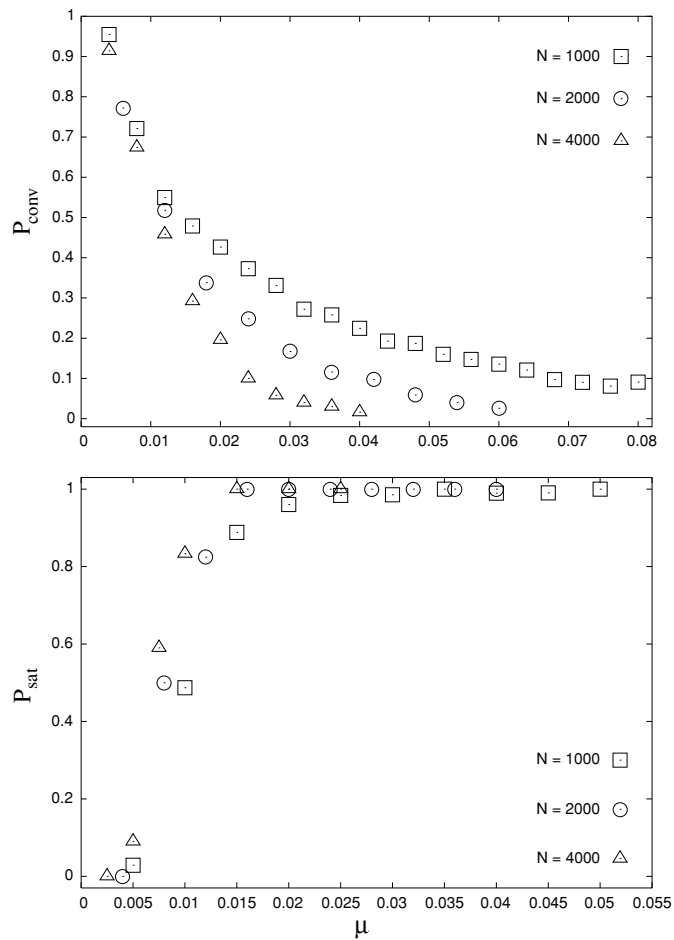


Figure 3. Convergence probability (P_{conv}) and satisfiability probability of a maximum spanning tree (P_{sat}) versus μ for random 3-SAT problems of different sizes with $p = 1/2$. Here the statistical errors are of order 0.01.

cases, edges of weight 1 have a considerable contribution and the weight distribution does not define a single characteristic weight. We think that it is these properties of $P(w)$ that allow the maximum-spanning-tree scenario to work. A clear difference between the two distributions in figure 4 is the presence of a delta peak at $w = 0$ for $\alpha = 4.2$. We found that this peak appears continuously around $\alpha \simeq 3.92$, where we expect the dynamical phase transition to happen.

We define the distance between two spanning trees as the number of uncommon edges. Then we take two randomly generated satisfiable spanning trees and obtain their distance d . We repeat this for a number of times and construct $P(d)$, the normalized number of times that the two spanning trees are at distance d of each other. The result has been shown in figure 5 for $\alpha = 3.8, 4, 4.2$. In these cases $P(d)$ is a Gaussian distribution and defines a characteristic distance which grows with α . Moreover, as the figure displays, by increasing α dispersion around the characteristic distance increases. Note that more distant factor graphs are more likely to have distant solutions. Therefore, we may conclude that as satisfiable spanning trees get away from each other the same happens in the solution space. These facts are consistent

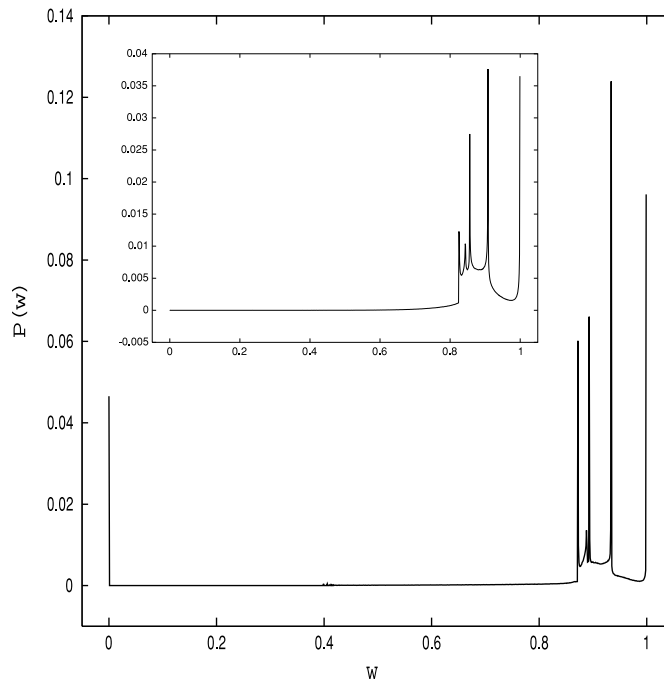


Figure 4. Weight distribution of edges in a random 3-SAT problem with $N = 1000$, $p = 1/2$ and $\alpha = 4.2, 3.9$ (inset) when μ is 0.025, 0.04, respectively. The data are result of averaging over 1000 realizations.

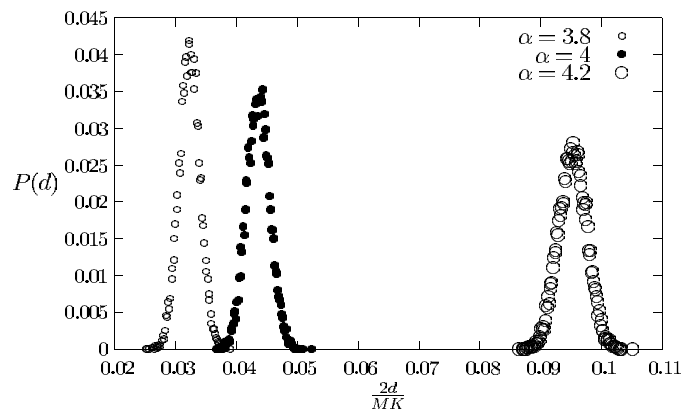


Figure 5. $P(d)$ for a single realization of random 3-SAT problem with $N = 1000$ and $p = 1/2$. For $\alpha = 3.8, 4, 4.2$ ($\mu = 0.055, 0.035, 0.025$, respectively) we have generated 10^4 pairs of satisfiable spanning trees.

with the present picture [3, 14] of the clustering phenomenon in the solution space. However, the presented data in figure 5 are not sufficient to detect the clustering phase transition in the solution space. To this end we should, probably, study the behaviour of the average distance and its dispersion for different problem sizes.

5. Conclusion

We showed that there is a correspondence between the solutions of a random K-SAT problem and those of spanning trees in the associated factor graph. This relation may be useful in finding more rigorous results for the structure of the solution space. The concept of null edges helped us to modify survey propagation algorithm and to construct satisfiable spanning trees. A satisfiable spanning tree has a number of solutions (in the same cluster of solutions) that can easily be obtained by local search algorithms. This provides us another way of extracting solutions of a random K-SAT problem. We used the modified algorithm to study organization of satisfiable spanning trees in the space of spanning trees. We found that by increasing the number of clauses per variable, the distance between satisfiable spanning trees increases. This finding supports the previous results on the clustering in the solution space.

In this paper we focused on the case $K = 3$ and $p = 1/2$, but we found that the algorithm works well for $K = 4, 5$ and other values of p (not represented here).

Acknowledgments

We would like to thank M Mezard who encouraged us to study this problem.

References

- [1] Kirkpatrick S and Selman B 1994 *Science* **264** 1297
- [2] Monasson R, Zecchina R, Kirkpatrick S, Selman B and Troyansky L 1999 *Nature* **400** 133
- [3] Mezard M 2003 *Science* **301** 1685
- [4] Mezard M, Parisi G and Zecchina R 2002 *Science* **297** 812
- [5] Monasson R and Zecchina R 1996 *Phys. Rev. Lett.* **76** 3881
- [6] Mezard M 2002 *Preprint cond-mat/0212448*
- [7] Mezard M, Mora T and Zecchina R 2005 *Phys. Rev. Lett.* **94** 197205
- [8] Monasson R and Zecchina R 1997 *Phys. Rev. E* **56** 1357
- [9] Mezard M and Zecchina R 2002 *Phys. Rev. E* **66** 056126
- [10] Braunstein A, Mezard M and Zecchina R 2005 *Random Struct. Algorithms* **27** 201
- [11] Barthel W, Hartmann A K and Weigt M 2003 *Phys. Rev. E* **67** 066104
- [12] Kschischang F R, Frey B J and Loeliger H-A 2002 *IEEE Trans. Inf. Theory* **47** 498
- [13] Montanari A, Parisi G and Ricci-Tersenghi F 2004 *J. Phys. A: Math. Gen.* **37** 2073
- [14] Mertens S, Mezard M and Zecchina R 2003 *Preprint cs/0309020*
- [15] Seitz S, Alava M and Orponen P 2005 *Preprint cond-mat/0501707*
- [16] Broder A 1989 *Proc. 30th IEEE Symp. on Foundations of Computer Science* p 442